$$\therefore a(R_1+R_2)=\frac{av^2}{g}[\sin 2\alpha+\sin 2(\alpha-\beta)]=\frac{2av^2}{g}\sin 2(\alpha-\beta)\cos\beta$$

$$=\frac{2av^2}{g}[\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)]\cos\beta$$

$$=\frac{a^4}{c^2}\left(\frac{\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}\right).$$

$$R_1R_2=\frac{v^4}{g^2}\sin 2\alpha\sin 2(\alpha-\beta)=\frac{4v^4}{g^2}\sin\alpha\cos\alpha\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$=\frac{a^4}{c^2}\frac{\sin\alpha\cos(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}.$$

$$\therefore a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}=\frac{a^4}{c^2}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

183. Proposed by MERTON T. GOODRICH, Dixfield, Maine.

What relations must exist between the quantities A, B, and C in the harmonic ratio $\frac{AB}{(A+B+C)(-C)}=-1$ so that they will be positive integers.

II. Solution by the PROPOSER.

Solving the given equation for A, we have $A=\frac{(B+C)C}{B-C}$. Since A, B, and C are positive, we see that $B-C$ must be positive. Since A is an integer, either $\frac{C}{B-C}=\frac{M}{N}=K$, or $\frac{B+C}{B-C}=\frac{M'}{N'}=K'$, where M and N, M' and N' are integers, and M prime to N, and M' prime to N'. From the first of these equations, $C=K(B-C)$. Since C and $B-C$ are positive, K must be positive. Solving this last equation for B, we have $B=\frac{(K+1)C}{K}=\frac{(M+N)C}{M}$. Since M and N are relatively prime, $M+N$ is prime to M. Hence, B being an integer, C is divisible by M. That is, $C=MD'=MND'/N=KD$, where $D=ND'$, and D' and hence D are positive integers.

Substituting KD for C in the expression for B, $B=(K+1)D=C+D$. Substituting KD for C and $(K+1)D$ for B in the expression for A, $A=(2K+1)KD=(2K+1)C$. Putting $\frac{M}{N}$ for K, $A=\frac{(2M+N)MD}{N^2}$. This tells us that if N is odd $D=N^2.\lambda$; but if N is even, $2M+N$ is even and then $D=\frac{N^2.\lambda}{2}$. Hence we have this set of relations: $A=(2K+1)C$, $B=C+D$,

$C=KD$, where K is a positive fraction in its lowest terms or a positive integer, and D is a positive integer as above determined.

The second condition is not independent of the first, because, if K' is substituted for $2K+1$ and $2D'$ for D, in the above set of relations, then we have

$$\frac{B+C}{B-C}=\frac{D'K'+D+'D'K'-D'}{D'K'+D'-D'K'+D'}=\frac{2D'K'}{2D'}=K',$$

which is the second condition. Hence the set of values found above includes all the possible relations which make A, B, and C positive integers. The values of A and B may be interchanged, and A, B, and C may each be multiplied by a common factor without changing the value of the original ratio.

Also solved by A. H. Holmes.

184. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that $\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\tan^{-1}\frac{1}{112^2}+\ldots$, where 2, 8, 30, 112..., is a recurring series with the recursion formula $u_n=4u_{n-1}-u_{n-2}$.

Solution by the PROPOSER.

If we reduce $\sqrt{3}$ to a continued fraction, we get for the convergents,

$$\frac{1}{1},\ \frac{2}{1},\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \frac{97}{56},\ \frac{265}{153},\ \frac{362}{209},\ \ldots$$

The alternate convergents,

$$\frac{1}{1},\ \frac{5}{3},\ \frac{19}{11},\ \frac{71}{41},\ \frac{265}{153},\ \ldots$$

are formed by taking the ratios of the corresponding terms of the two recurring series

$$1,\ 5,\ 19,\ 71,\ 265,\ \ldots$$
$$1,\ 3,\ 11,\ 41,\ 153,\ \ldots$$

both having the same scale of relation $u_n=4u_{n-1}-u_{n-2}$.

We find by the usual methods that the nth terms of the two series are

$$\frac{\alpha^n-\beta^n}{\alpha-\beta}+\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n+u_{n-1},\text{ and }\frac{\alpha^n-\beta^n}{\alpha-\beta}-\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n-u_{n-1},$$